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BASIC GEOMETRICAL SINGULARITIES IN PLANE-ELASTICITY AND PLATE-BENDING PROBLEMS

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Abstract—Volterra's "distorsioni" can be considered as the basic geometrical singularities for slice and plate problems. They are characterized by the related discontinuity of the displacement. Because any dislocation can be interpreted as a dipole of two disclinations, the latter represents the ancestral geometrical singularities, entirely equivalent to the single force as the ancestral statical singularity. The aim of the paper is to systematically classify Volterra's "distorsioni," demonstrating their significance for plane-elasticity and plate-bending problems, and discussing the related displacement fields. Methodical examination and illustration by many schematic drawings reveal the analogies (slab analogy) between dislocations and disclinations, respectively, and the corresponding statical singularities as the single force and its derivatives.

1. INTRODUCTION

Michell[1] (1899), Weingarten[2] (1901) and Timpe[3] (1905) were the first to study the possibility that an elastic continuum might be subjected to internal stresses, in spite of the fact that there are neither body forces nor surface forces, and that St. Venant's equation of compatibility is fulfilled. This phenomenon was studied in more detail in a series of notes by Volterra, starting from the question whether such a state of internal stress could occur in a simply-connected elastic body; it can be easily proven that the answer is no (again supposing that St. Venant's equation is fulfilled†). For a multiply-connected body there is, however, the physical possibility of internal stresses without any forces acting (or other sources†) because then the displacement is no longer necessarily single-valued. Now the multiply-connected region can be reduced to a simply-connected region by means of a system of barriers, each of which is characterized by six constants describing the discontinuity of the displacement when crossing the barrier, and the uniqueness of solution is secured if, and only if, these six constants of every barrier are given.

It was Volterra's great merit to discuss the different types of possible discontinuities of the displacement in a multiply-connected elastic body in detail, and to illustrate their geometrical and physical meanings in a very conspicuous way. He published an excellent comprehensive account of the theory (with some improvements by Cesaro[5]) in 1907[6]; a survey of the main aspects was later given by Love[7] (who suggested the name "dislocations" instead of Volterra's "distorsioni").

Now, since 1907, much research has been done in dislocation theory, mainly due to the discovery that the physical properties of crystal structures are largely influenced by the presence of lattice defects; therefore, we shall give only a short introduction to Volterra's "distorsioni." The aim of this paper consists in demonstrating that they represent the basic geometrical singularities in plane-elasticity and plate-deflection problems, entirely equivalent to the basic statical singularities as the single force and its derivatives. Statical and geometrical singularities are of completely equal status as far as their mechanical importance for the solution of the fundamental problems of elastostatics is concerned. Nevertheless, most readers will be far less familiar with

† Implying that there are no sources for an "extra strain," i.e. a prescribed nonelastic strain (as, e.g. caused by thermal stresses, plastic deformation, electrical and magnetical processes, etc.). Strangely enough, Volterra obviously did not recognize the possibility of the existence of such extra strains. To our knowledge, the first to have clearly understood the important mechanical difference between elastic strain and nonelastic strain was Foeppl in 1907 ([4], pp. 293–311).

dislocations and disclinations, respectively, than with forces; to help to change this is the main purpose of this paper.

From the extensive literature on dislocation theory we only chose those books and papers that are of special importance in our context; a thorough list of references was compiled, e.g. by Teodosiu[8].

2. VOLTERRA'S SIX ORDERS OF "DISTORSIONI"

Following Volterra's suggestion[6] let us consider the simplest multiply-connected body, i.e. a cylinder with a central hole. The cylinder, free of body forces, surface forces, or any initial stress, is rendered singly-connected by being cut open along a radial plane; after the cut, the two surfaces A_1 and A_2 themselves [see Fig. 1(b)] are supposed to be completely rigid, the whole of the rest of the cylinder behaving elastically. For our models, we chose cylinders made of rubber, to whose two cut surfaces two thin sheets of steel were glued [see Fig. 1(a)]; Volterra himself already had rubber models made for demonstration purposes[6, 9]. Now, according to Volterra, six different types of manipulations are possible: for each we displace the two surfaces A_1 and A_2 relative to each other without deforming them, if necessary remove any overlapping material (fill up any gap with the same elastic material), and then cement the surfaces to each other again (to the elastic material which was filled in). In each case the result in the cylinder, now rendered doubly connected again, will be a state of internal stress. Its components vary continuously across the surface A_1 and A_2 as in an elastic body in its "natural" state; the displacement, however, is now no longer single-valued as will be demonstrated below.

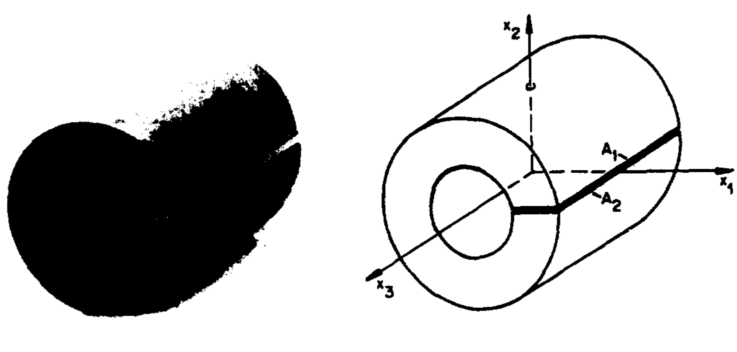
2.1. *Translation of the cut surfaces ("dislocations") and rotation of the cut surfaces ("disclinations")*

For the described displacing of A_2 with respect to A_1, two different kinds have to be distinguished: the first kind (I) is a pure translation of the two surfaces, i.e. they are still parallel in the deformed state, and in the second kind (II) they have experienced a rotation relative to each other. In the following description we refer to the coordinates x_1, x_2, x_3 of Fig. 1(b).

(I) *Translation of the cut surfaces relative to each other.* This way of deforming the cut cylinder can be described by a vector b_i (Burgers vector[10]) pointing from A_1 to A_2.

(I-1) A translation with the vector $b_i = (b_1, 0, 0)$ is produced by pulling the cut surface A_2 away from A_1 in the $+x_1$-direction.

(I-2) A translation with the vector $b_i = (0, (\pm) b_2, 0)$ is produced by cutting out a square block of thickness b_2 (by inserting a square block of thickness b_2 of the same elastic material) and then cementing the two surfaces A_2 and A_1 to each other (to the square block inserted).



(a) (b)

Fig. 1. The original cut cylinder: (a) rubber model; (b) cut surfaces A_1, A_2, and coordinates.

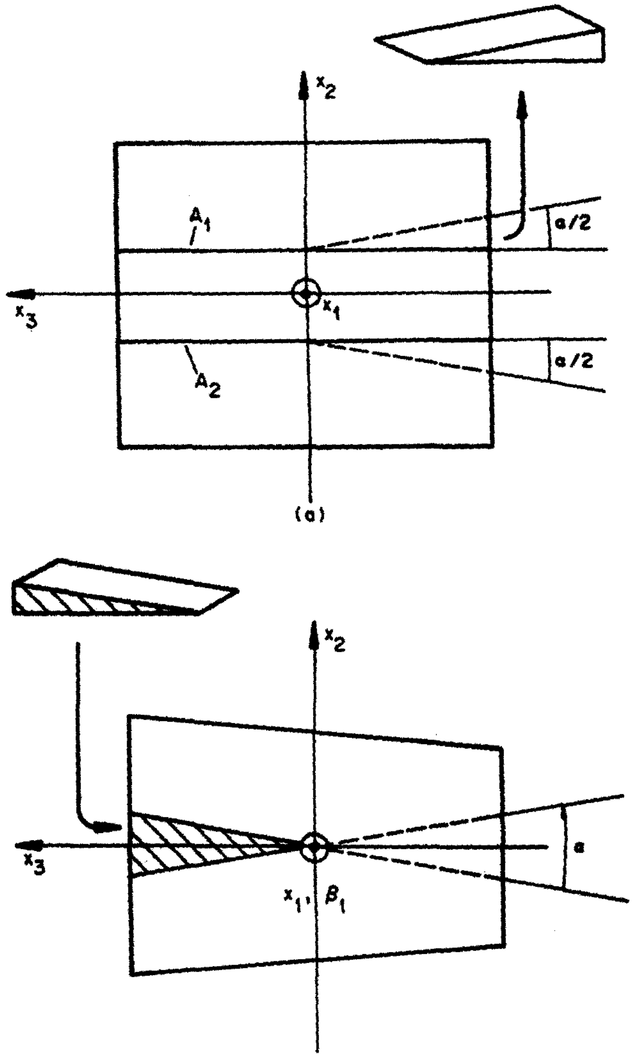


Fig. 2. Producing the rotation type (II-1): (a) open the cut cylinder and remove two wedges of angle $\alpha/2$ from the back part ($x_3 < 0$); (b) insert them into the front part ($x_3 > 0$), close the cylinder, and glue all contact surfaces together.

(I-3) A translation with the vector $b_i = (0, 0, b_3)$ is produced by pulling the surface A_2 against A_1 in the $+x_3$-direction.

(II) *Rotatioin of the cut surfaces relative to each other.* This way of deforming the cut cylinder can be described by a vector β_i (Frank vector: see Dewit[11]) defined in the following way: with the rotation angle α being counted from A_1 to A_2 [cp. Fig. 2 and 3(b)], the magnitude of β_i is given by α, its orientation is vertical to the plane defined by the two sides of α, and its direction is connected to that one of α by the right-hand rule.

(II-1) A rotation with the vector $\beta_i = (\beta_1, 0, 0)$ is produced by rotating A_2 with respect to A_1 through the angle α in the x_2x_3-plane; hereby, because any translation is excluded, the center lines of A_2 and A_1 in the x_1-direction still lie on top of one another in the deformed state, and the two halves of A_2 and A_1 with $x_3 < 0$ penetrate each other.

(II-2) A rotation with the vector $\beta_i = (0, \beta_2, 0)$ is produced by rotating A_2 with respect to A_1 through the angle α in the x_1x_3-plane.

(II-3) A rotation with the vector $\beta_i = (0, 0, (\pm)\beta_3)$ is produced by cutting out a wedge (by inserting a wedge of the same elastic material) of angle α in the x_1x_2-plane and then cementing the two surfaces A_2 and A_1 together (to the wedge inserted).

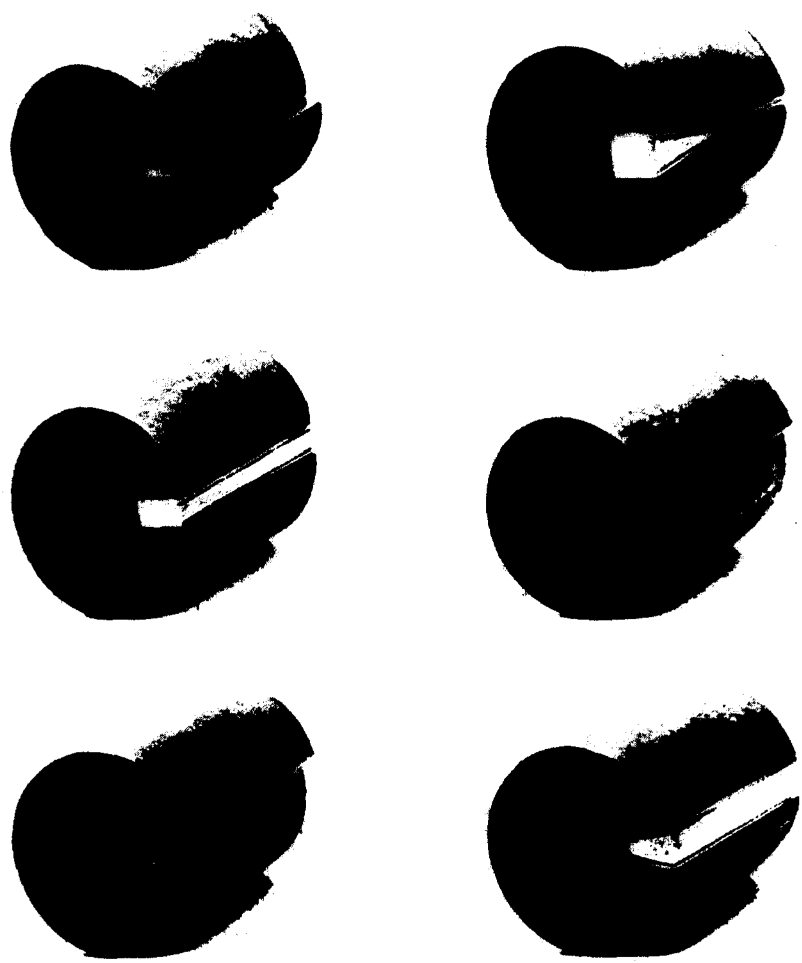


Fig. 3a. Rubber models of Volterra's dislocations (I) and disclinations (II).

Photographs of the described six types are shown in Fig. 3(a), taken from rubber models (cp. Volterra[6, 9]), and the corresponding schematic drawings in Fig. 3(b) (cp. Dewit[11] and Harris[12]). Two "technical" remarks shall be made to Fig. 3: — 1. What is hardly to be recognized from the photographs and was suppressed on purpose in the drawings, are the deformations of the surfaces of the tubes, including those of the end surfaces; they are of no special interest in our context. — 2. Whereas in the other four cases the vectors b_i and β_i, respectively, have positive directions (according to our above sign convention), we preferred for the two types (I-2) and (II-3), respectively, to insert material instead of removing it, and thus to create vectors b_2 and β_3, respectively, opposite to the directions of x_2 and x_3, respectively, because then the photographs and drawings of these two types are easier to produce and to understand.

For a long period, there was no generally accepted agreement in the literature on the names for the described types. Volterra himself[6, 9] called them "distorsioni" and simply counted them through: "first order" for the type (I-1), "second order" for (I-2) and so on up to the "sixth order" for (II-3). It was Love[7] who ventured to call them "dislocations." Frank suggested to restrict the name "dislocations" for the types (I-1)–(I-3) and recommended the name "disinclinations" for the types (II-1)–(II-3), which was later changed to "disclinations." Analogously to the two types of dislocations, namely the "*edge*" *dislocation* [(I-1), (I-2)] and the "*screw*" *dislocation* [(I-3)], Nabarro[13] classified disclinations as edge types [(II-1), (II-2)] and screw types [(II-3)]. The Stuttgart school (e.g. [14, 15]) however, preferred the name "*wedge*" *disclination* for (II-3), and Eshelby (see [16]) suggested calling (II-1) and (II-2) "*twist*"

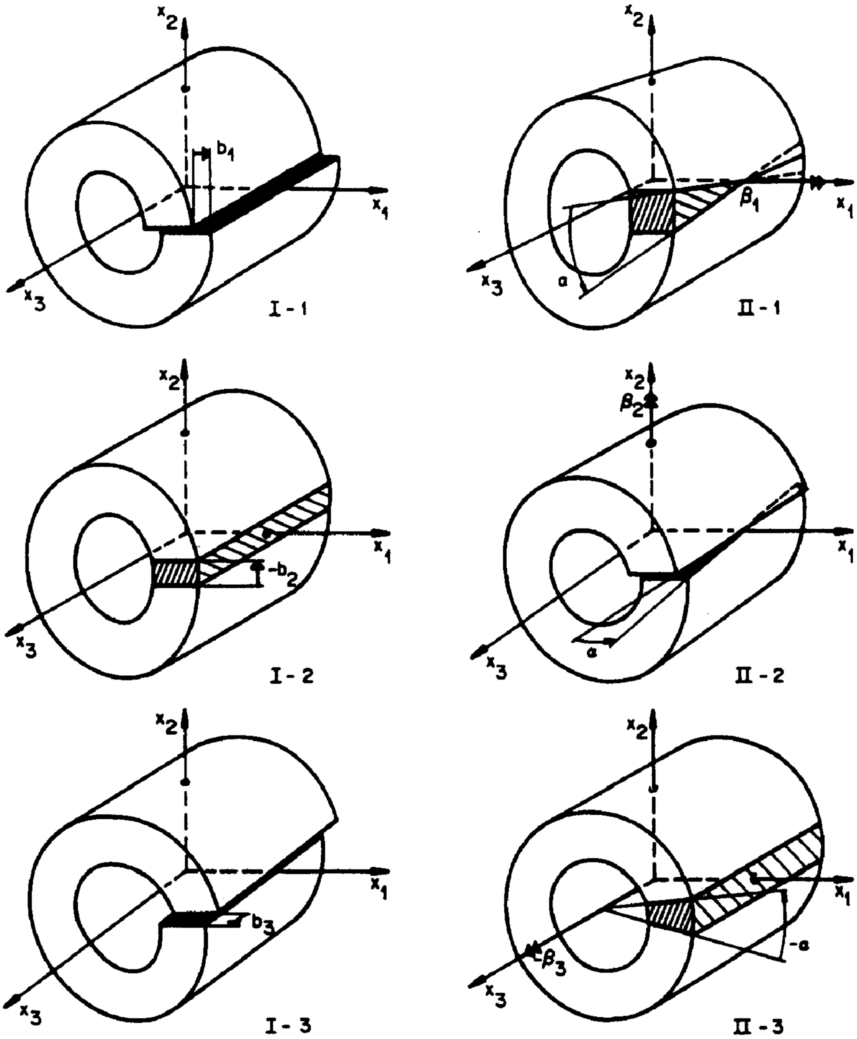


Fig. 3b. Volterra's dislocations (I) and disclinations (II), and the vectors b_i (I) and β_i (II) describing them.

disclination rather than "edge" disclination. — The *italicized names* will be used in this paper [see Fig. 3(a)].

2.2. *Discontinuity of the displacement*

The characteristic common to all six types of Volterra's dislocations and disclinations, respectively, is the fact that whereas stress and strain in the deformed body are still single-valued, continuous, and satisfying the equations of equilibrium and compatibility, the displacement is no longer single-valued. In order to understand this effect more clearly, let us come back to the original cut cylinder and consider two points P_1 and P_2 of the cut surfaces A_1 and A_2, respectively, which are directly opposite neighbors. They are connected by the circular line S which is, in the original undeformed state of the cylinder, closed and lying in a plane parallel to the x_1x_2-plane; S starts at P_1 and ends at P_2, i.e. its orientation is defined in a right-handed sense relative to the x_3-axis [see Fig. 4(a)]. S is called the Burgers circuit ([10, 17]).

Now, if we follow the same circuit S after the cylinder has experienced a small deformation of type (I), we have to add a vector $(-b_i)$

$$(-b_i) + \oint_{(S)} (\partial u_i/\partial s)\, \mathrm{d}s = 0 \qquad (i = 1, 2, 3) \tag{2.1}$$

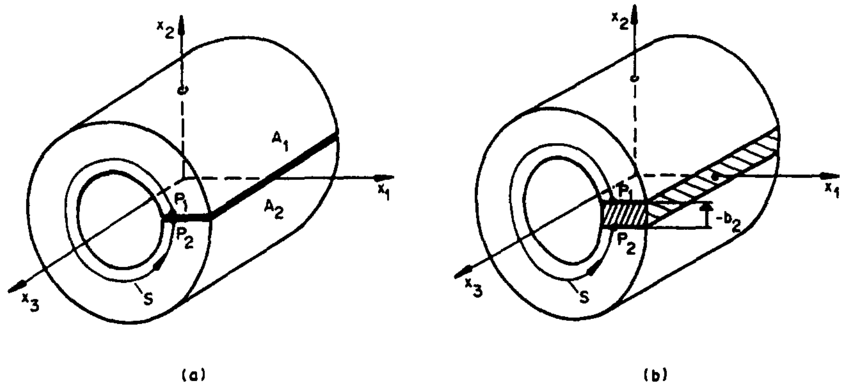


Fig. 4. Burgers circuit S and Burgers vector b_i: (a) original cut cylinder ($b_i = 0$); (b) edge dislocation [$b_i = (0, -b_2, 0)$].

in order to arrive at the starting point P_1 [see Fig. 4(b)]. b_i is called the Burgers vector and pointing, in the above definition, from P_1 to P_2. Thus the discontinuity $\Delta u_i^{(I)}$ of the displacement of P_2 with respect to its original neighbour P_1 is

$$\Delta u_i^{(I)} = b_i \qquad (i = 1, 2, 3). \tag{2.2}$$

Following the circuit S after a small deformation of type (II), we obtain a displacement vector $\Delta u_i^{(II)}$ calculated according to

$$\Delta u_i^{(II)} = e_{ghi}\beta_g(x_h - \mathring{x}_h) \qquad (i, g, h = 1, 2, 3) \tag{2.3a}$$

where e_{ghi} is the permutation tensor (Levi–Civita tensor), β_g the rotation vector (Frank vector) and x_h the radius vector of P_1 and P_2, respectively, in the undeformed state. $\mathring{x}_h$ is the radius vector of the corresponding rotation center, i.e. of the point(s) characterized by $\Delta u_i^{(II)} = 0$; according to our definition [cp. Fig. 3(b)] $\mathring{x}_h$ is

$$\begin{aligned} \mathring{x}_h &= (r_1 \leqq \mathring{x}_1 \leqq r_2, 0, 0) && \text{for (II-1)}, \\ \mathring{x}_h &= ((r_1 + r_2)/2, 0, 0) && \text{for (II-2)}, \\ \mathring{x}_h &= (0, 0, -l/2 \leqq \mathring{x}_3 \leqq l/2) && \text{for (II-3)} \end{aligned} \tag{2.4}$$

with l being the length of the cylinder (in its original state), r_2 its radius, and r_1 the radius of the central hole.

With eqn (2.4), eqn (2.3a) simply yields

$$\Delta u_i^{(II)} = e_{ghi}\beta_g x_h \tag{2.3b}$$

(apart from an unimportant additional constant $\beta_2(r_1 + r_2)/2$ in the expression for $\Delta u_3^{(II)}$).

For any combination of deformations of the type (I) and the type (II) we get by superimposing

$$\Delta u_i = \oint_{(S)} (\partial u_i/\partial s)\,ds = \Delta u_i^{(I)} + \Delta u_i^{(II)} = b_i + e_{ghi}\beta_g x_h, \quad (g, h, i = 1,2,3), \tag{2.5a}$$

or written at full length

$$\begin{aligned} \Delta u_1 &= b_1 + 0\cdot x_1 - \beta_3 x_2 + \beta_2 x_3, \\ \Delta u_2 &= b_2 + \beta_3 x_1 + 0\cdot x_2 - \beta_1 x_3, \\ \Delta u_3 &= b_3 - \beta_2 x_1 + \beta_1 x_2 + 0\cdot x_3. \end{aligned} \tag{2.5b}$$

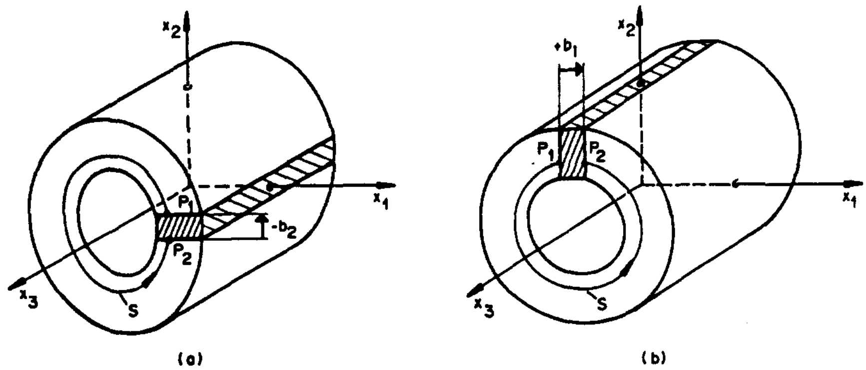


Fig. 5. To the equivalence of the two edge dislocations (a) type (I-2) and (b) type (I-1) of Fig. 3.

This discontinuity of the displacement, jumping for Δu_i with every Burgers circuit S, remains unaltered if we let the inner radius r_1 tend to zero, the outer radius r_2 and the length l to infinity

$$r_1 \to 0, \qquad r_2 \to \infty \quad \text{and} \quad 1 \to \infty; \tag{2.6}$$

we thus arrive at an elastic continuum with a singularity line along the x_3-axis, i.e. a dislocation line for the types (I-1)–(I-3) and a disclination line for the types (II-1)–(II-3).

A closer look at Fig. 3(b) and eqn (2.5) shows at once that two pairs of the six types of dislocations and disclinations, respectively, differ only by their spatial orientation as far as the discontinuity Δu_i of the displacement is concerned: if we keep the axes x_1 and x_2 fixed and let the cylinder rotate through $+90°$ around x_3, the dislocation type (I-2) with vector $-b_2$ changes into the type (I-1) with vector $+b_1$, as shown in Fig. 5. Analogously the disclination type (II-2) with vector $+\beta_2$ changes into the type (II-1) with vector $-\beta_1$. Thus, instead of six there are only four types that are really different from the physical point of view, i.e. from the discontinuity Δu_i of the displacement, namely (I-1) and (I-2), and (I-3) on the one hand, and (II-1) and (II-2), (II-3) on the other hand.

2.3. *Generalized dislocations and disclinations in elastic continua*

If we generalize the procedure that Volterra applied to a hollow cylinder, the most general way of creating a linear defect of Volterra type consists in the following series of operations[18, 19].

1. Cut the elastic medium along an arbitrary surface A, not necessarily plane, bounded by a line L.
2. Displace the two lips A_1 and A_2 of the cut relative to each other without deforming them.
3. Remove any overlapping material (fill any gap with the same elastic material).
4. Stick the material along the surfaces A_1 and A_2, and remove the external forces applied during the operation.

The internal stresses thus created produce a line discontinuity along L. They vary continuously across the surfaces A_1 and A_2, as well as the strain components, whereas the displacement is discontinuous across A.

A still more general case is represented by the so-called Somigliana "dislocation." Somigliana[20] showed that the conditions of elastic equilibrium may be satisfied as long as only the tractions across the cut are continuous, while the tangential components of stress may be discontinuous, i.e. St. Venant's equation of compatibility is no longer fulfilled along A. Therefore step 2 of the above scheme can be generalized in the following way:

2′. Displace the two lips A_1 and A_2 of the cut relative to each other whereby they may be deformed.

Thus to each pair of points P_1 and P_2 adjacent to one another on opposite sides A_1 and A_2 of the cut A is assigned a relative displacement vector Δu_i.

Obviously, Volterra's dislocations and disclinations, respectively, are particular cases of this general type of "dislocations" suggested by Somigliana: if Δu_i has a constant value all over A [cp. eqn (2.2)], we have Volterra's dislocations (I-1)–(I-3), and if Δu_i takes the form $e_{ghi}\beta_g x_h$ with β_g being constant [cp. eqn (2.3b)], we have Volterra's disclinations (II-1)–(II-3).

2.4. *Dislocations as dipoles of disclinations*

Any dislocation can be interpreted as a dipole of two opposite disclinations[1] and [13, 21, 22]. Because this possibility is essential for our later considerations, it shall be demonstrated here for Volterra's models.

We define a general disclination dipole in the following way, which is completely analogous to the definition of the force dipole by Kroener[23]: we apply a disclination β_j, with all three components β_1, β_2, β_3, respectively, pointing into the negative direction of the axes x_1, x_2, x_3, respectively, at a point $\bar{x}_i$ of the elastic continuum, and in a neighboring point $\bar{x}_i + \delta\bar{x}_i$ we apply a second disclination β_j^* of the same magnitude, but of opposite direction to the original one. If we now let the distance vector $\delta\bar{x}_i$ tend to zero, while keeping the amount of $\delta\bar{x}_i \cdot \beta_j$ constant

$$\beta_{ij} = \lim_{\delta\bar{x}_i \to 0} \delta\bar{x}_i \cdot \beta_j = \text{const} \qquad (i, j = 1, 2, 3), \tag{2.7}$$

we obtain a disclination dipole β_{ij}.

Let us for example consider the wedge disclination (II-3) of Fig. 3 with the radius r_1 of the central hole now tending to zero and the length l tending to infinity. We first insert a wedge of angle α and thus create a wedge disclination line L_1 along the x_3-axis, described by the vector β_3 pointing into the negative x_3-direction; then we remove a wedge of the same angle α in a neighboring position $\delta\bar{x}_i = (\delta\bar{x}_1, 0, 0)$ and thus create a second wedge disclination line L_2, also along the x_3-axis but at a distance $\delta\bar{x}_1$ from L_1, with the vector β_3^* opposite to β_3 [in accordance with Fig. 6(b); in Fig. 6(a), however, the order of the two operations was permuted, since then the drawing is easier to understand]. The corresponding wedge disclination dipole β_{13} according to eqn (2.7) obviously represents an edge dislocation of type (I-2) with the Burgers vector b_2 pointing into the $-x_2$-direction. Analogously, a wedge disclination dipole β_{23} can be interpreted as an edge dislocation with positive Burgers vector b_1; the remaining type of a wedge disclination dipole, namely β_{33}, represents no "dipole" at all, since first inserting a wedge and then removing it in a neighboring position $\delta\bar{x}_i = (0, 0, \delta\bar{x}_3)$ would simply recreate the original cylinder (for $l \to \infty$). Thus, generally speaking, any edge dislocation

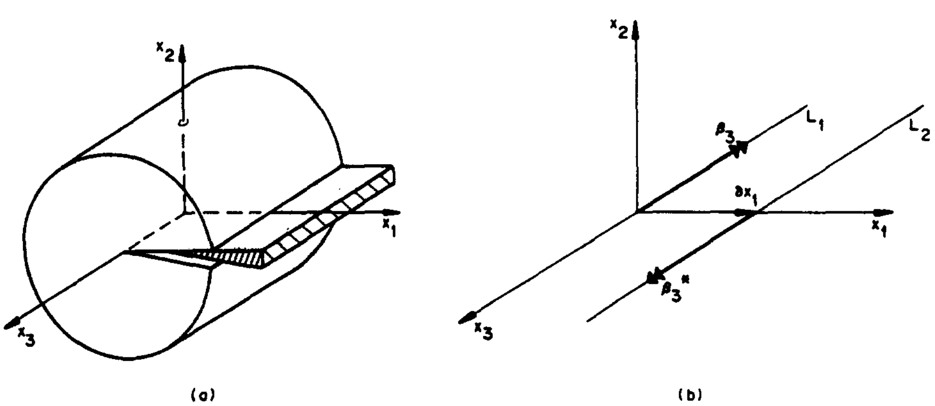


Fig. 6. Two wedge disclinations with opposite vectors β_3 and β_3^*: (a) model according to Volterra [cp. type (II-3) of Fig. 3(b)]; (b) disclination lines L_1, L_2, and distance vector δx_1.

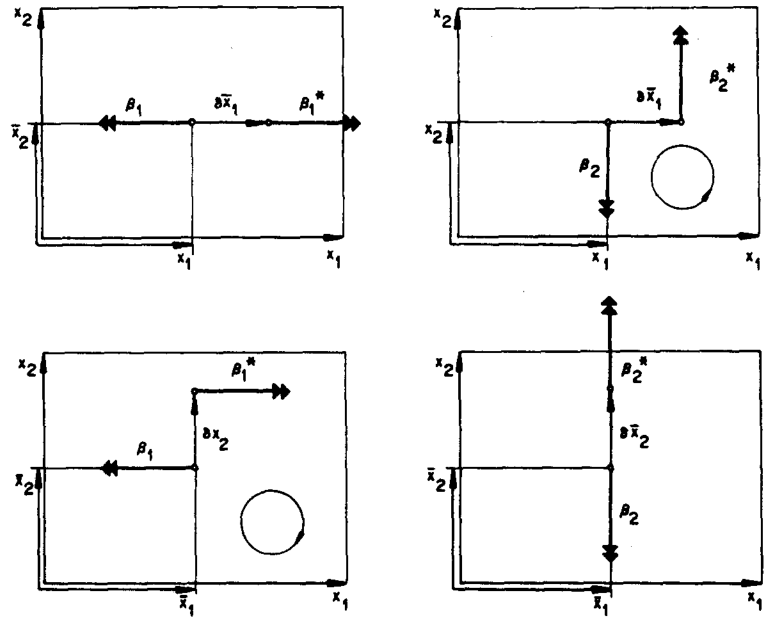


Fig. 7. To the definition of the twist disclination dipole β_{ij} $(i, j = 1, 2)$.

can be interpreted as a dipole of a two wedge disclinations, the vector of which (given by $\delta\bar{x}_i$) has been rotated through $-90°$ around the $+x_3$-axis (cp. [24]).

On the other hand, an edge dislocation can also be interpreted as a dipole of two opposite twist disclinations β_1 and β_2, respectively (see, e.g. [14]); again the Burgers vector, b_1 and b_2, respectively, is obtained by rotating the corresponding dipole vector through $-90°$. The twist disclination dipole β_{32}, with the distance vector $\delta\bar{x}_i$ falling completely into the x_3-direction, represents an edge dislocation $-b_1$; and analogously β_{31} can be interpreted as an edge dislocation $+b_2$. These two cases are of no further interest for our later considerations.

Thus an edge dislocation is equivalent either to a wedge disclination dipole β_{13} and β_{23}, respectively, or to a twist disclination dipole β_{31} and β_{32}, respectively. The second dislocation type (I-3), the screw dislocation with Burgers vector b_3, is equivalent to a twist disclination dipole β_{12} and β_{21}, respectively: if the distance vector $\delta\bar{x}_i$ falls completely into the x_1-direction, i.e. for $\delta\bar{x}_i = (\delta\bar{x}_1, 0, 0)$, the corresponding twist disclination dipole β_{12} represents a screw dislocation $+b_3$, i.e. with the Burgers vector pointing into the $+x_3$-direction; and similarly β_{21} can be interpreted as a screw dislocation $-b_3$.

It should be mentioned that the twist disclination dipole $\beta_{ij}(i, j = 1, 2)$ is completely analogous to the force dipole r_{ij} $(i, j = 1, 2)$ in plane stress problems (cp. [7]): in the same sense as r_{12} and r_{21} represent double forces with nonvanishing moment, β_{12} and β_{21} represent double twist disclinations with nonvanishing Burgers vector b_3; the rotation direction of β_{12}, positive in the mathematical sense, is identical with the orientation of the corresponding right-handed Burgers circuit for the screw dislocation b_3 (cp. Figs. 4 and 7). Like an antisymmetric force dipole $r_{ij} = e_{ij}r/2$, the deformation field of which represents the arithmetic mean of two double forces r_{21} and r_{12} with the same moment, i.e. $r_{21} = -r_{12}$, is said to be a rotation center with moment r, an antisymmetric twist disclination dipole in the x_1x_2-plane

$$\beta_{ij} = e_{ij}b_3/2 \qquad (i, j = 1, 2) \tag{2.8}$$

(with the permutation tensor e_{ij}: $e_{12} = -e_{21} = 1$, $e_{11} = e_{22} = 0$) represents a screw dislocation center with Burgers vector b_3 pointing into the positive x_3-direction.

The remaining two components β_{11} and β_{22} represent twist disclination dipoles without Burgers vector b_3 and are completely analogous to double forces r_{11} and r_{22}

Table 1. The nine disclination dipoles β_{ij} and their interpretation as dislocations

disclination dipole	dislocation type
wedge: β_{13}	edge: $-b_2$
" : β_{23}	" : $+b_1$
" : (β_{33})	-.-.-
twist: β_{31}	edge: $+b_2$
" : β_{32}	" : $-b_1$
twist: β_{12}	screw: $+b_3$
" : β_{21}	" : $-b_3$
" : β_{11}	-----
" : β_{22}	-----

with vanishing moment. β_{11} and β_{22} cannot be interpreted as dislocations (cp. Table 1); they are, however, of the same importance for the plate problem (cp. eqn (3.25) with $\beta_{ij} = d_{ij}$) as r_{11} and r_{22} for the slice problem. The different types of disclination dipoles are listed in Table 1.

Of special interest for our later considerations are the wedge disclination dipoles β_{13} and β_{23} and the last mentioned twist disclination dipoles β_{ij} $(i, j = 1, 2)$. To make the relationship between the disclination dipoles β_{ij} and the dislocations b_i at once evident, we reformulate eqns (2.2) and (2.3), respectively, which describe the influence of b_i and β_{ij}, respectively, on the corresponding jump Δu_i of the displacement:

(I) Instead of eqn (2.2) we write

$$\Delta u_i = \delta_{ij} b_j \qquad (i, j = 1, 2, 3) \tag{2.9a}$$

or using the practical form suggested by Heise[24],

$$\begin{aligned} &\Delta u_i = (\Delta u b)_{i,j} \cdot b_j \\ &\text{with} \quad (\Delta u b)_{i,j} = \delta_{ij} \end{aligned} \qquad (i, j = 1, 2, 3) \tag{2.9b}$$

with the identity tensor $\delta_{ij}(\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j)$ and the influence function $(\Delta u b)_{i,j}$ depicting the jump Δu_i caused by the dislocation b_j.

(II) Analogously, eqn (2.3b) take the form

$$\begin{aligned} &\Delta u_i = (\Delta u \beta)_{i,j} \cdot \beta_j \\ &\text{with} \quad (\Delta u \beta)_{i,j} = e_{ijh} x_h \end{aligned} \qquad (i, j, h = 1, 2, 3), \tag{2.10a}$$

or, more generally, namely if the disclination β_j is no longer applied at the origin of the $x_1x_2x_3$-system, but at an arbitrary point $\bar{x}_h \neq 0$

$$(\Delta u \beta)_{i,j} = e_{ijh}(x_h - \bar{x}_h) \qquad (i, j, h = 1, 2, 3). \tag{2.10b}$$

Now, it follows from the definition (2.7) that we obtain the influence function $(\Delta u \beta)_{i,jk}$, depicting the jump Δu_i caused by the disclination dipole β_{jk}, by differentiating $(\Delta u \beta)_{i,k}$ with respect to $\bar{x}_h$:

$$(\Delta u \beta)_{i,jk} = \bar{\partial}_j(\Delta u \beta)_{i,k} \qquad (i, j, k = 1, 2, 3) \tag{2.11a}$$

which yields, with eqn (2.10b) and because of $\bar{\partial}_j\bar{x}_h = \delta_{jh}$,

$$\begin{aligned}\Delta u_i &= (\Delta u\beta)_{i,jk}\cdot\beta_{jk} \\ \text{with} \quad (\Delta u\beta)_{i,jk} &= e_{ijk}\end{aligned} \qquad (i, j, k = 1, 2, 3). \tag{2.11b}$$

The results of eqn (2.11b) for the different values of j, k = 1, 2, 3 had already been listed in Table 1:

(a) The wedge disclination "dipole" β_{33} (no "real" dipole) and the twist disclination dipoles β_{11}, β_{22} cause no jump Δu_i at all: $(\Delta u\beta)_{i,jj} = 0$.
(b) The wedge disclination dipole β_{13} and β_{23}, respectively, causes the jump $\Delta u_2 = -\beta_{13}$ and $\Delta u_1 = +\beta_{23}$, respectively, i.e. the same jump as caused by the edge dislocation $-b_2$ and $+b_1$, respectively.
(c) The same relationship is valid, apart from the sign, for the twist disclination dipole β_{31} and β_{32}, respectively.
(d) The twist disclination dipole β_{12} and β_{21} causes the jump $\Delta u_3 = +\beta_{12}$ and $\Delta u_3 = -\beta_{21}$, respectively, i.e. the same jump as caused by the screw dislocation $+b_3$ and $-b_3$, respectively.

3. DISLOCATIONS AND DISCLINATIONS AS THE BASIC GEOMETRICAL SINGULARITIES IN PLANE-ELASTICITY AND PLATE-BENDING PROBLEMS

The character of Volterra's dislocations and disclinations, respectively, is essentially geometrical: according to eqn (2.5) they are defined by the jump Δu_i of the displacement which is caused by them; this relationship is analogous to that between the single force as the basic statical singularity and the stress caused by it.

Further investigations into this equivalency will be found in a forthcoming paper of the author[25]. In our context here we only give a short example. Let us consider a slice of thickness h with an edge dislocation vector $b_i = (b_1, b_2)$, as shown in Fig. 8. According to eqn (2.1), the relation between b_i and $\partial u_i/\partial s$ for any Burgers circuit S is given by

$$-b_i + \oint_{(S)} (\partial u_i/\partial s) = 0 \qquad (i = 1, 2), \tag{3.1a}$$

or with the tangential vector t_j

$$-b_i + \oint_{(S)} t_j\partial_j u_i \,\mathrm{d}s = 0 \qquad (i, j = 1, 2). \tag{3.1b}$$

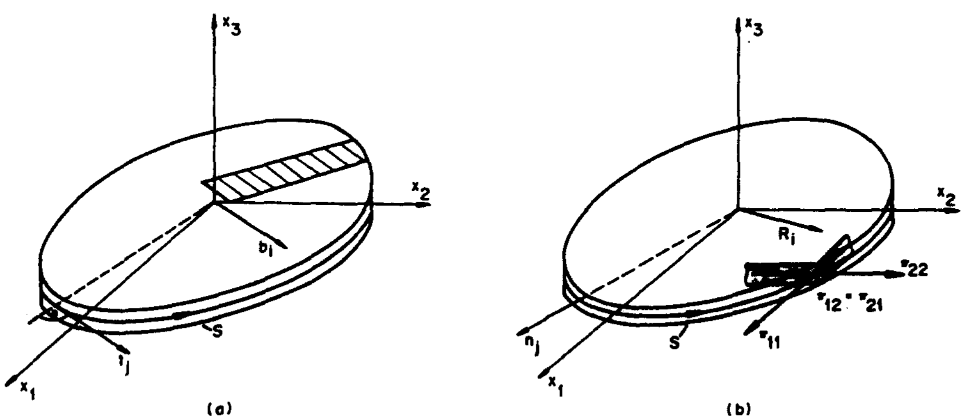


Fig. 8. Slice subjected to (a) an edge dislocation b_i; (b) a single force R_i (i = 1, 2).

If the slice is now subjected to the basic statical singularity, i.e. the single force $R_i = (R_1, R_2)$, instead of b_i (cp. Fig. 8), the condition of equilibrium yields for any closed circuit S

$$+R_i + \oint_{(S)} n_j \Pi_{ji} \, \mathrm{d}s = 0 \qquad (i, j, = 1, 2), \tag{3.2a}$$

with n_j being the normal vector, or because of $n_j = e_{ja} t_a$

$$+R_i + \oint_{(S)} t_j e_{bj} \Pi_{ib} \, \mathrm{d}s = 0 \qquad (b, i, j = 1, 2). \tag{3.2b}$$

The comparison of eqns (3.1) and (3.2) shows at once that the relationship between the geometrical quantities $-b_i$ and $t_j \partial_j u_i$ is analogous to that between the statical quantities $+R_i$ and $t_j e_{bj} \Pi_{ib}$.

3.1. *Slice problem and plate problem*

In the following, we let the inner radius r_1 of Volterra's hollow cylinder tend to zero, its outer radius r_2 to infinity, and keep its length l at a constant, sufficiently small value h [cp. eqn (2.6)]

$$r_1 \to 0, \qquad r_2 \to \infty, \qquad l = h; \tag{3.3}$$

we thus arrive at an infinite plate or an infinite slice, respectively,† of thickness h with a singularity line along the x_3-axis. The corresponding six types of dislocations and disclinations are composed in Fig. 9 (cp. Fig. 3):

From Fig. 9 it is obvious that two cases have to be distinguished when a plane elastic continuum is subjected to a dislocation (I) or a disclination (II), respectively:

Case 1—slice problem. The edge dislocation (I-1), (I-2) and the wedge disclination (II-3) cause jumps Δu_1 and/or Δu_2 of the displacement vector components in the middle plane $x_1 x_2$ for every Burgers circuit S, but none in the x_3-direction:

$$\Delta u_3 = 0. \tag{3.4-1}$$

Case 2—plate problem. On the contrary, the remaining three types, namely the screw dislocation (I-3) and the twist disclination (II-1), (II-2) are characterized by

$$\Delta u_3 \neq 0. \tag{3.4-2}$$

Thus if we are only interested in the elastic state of deformation in the middle plane $x_1 x_2$ itself, i.e. if we set $x_3 = 0$, eqn (2.5b) is split into the two separate equations

$$\begin{aligned} \Delta u_1 &= b_1 + 0 \cdot x_1 - \beta_3 x_2, \\ \Delta u_2 &= b_2 + \beta_3 x_1 + 0 \cdot x_2 \end{aligned} \tag{3.5-1}$$

for the slice problem, and

$$\Delta u_3 = b_3 - \beta_2 x_1 + \beta_1 x_2 \tag{3.5-2}$$

for the plate problem. Therefore, the edge dislocation with Burgers vector b_1, b_2 and the wedge disclination with Frank vector β_3 are the basic (meaning simplest) geo-

† The names plate (slab) and slice are used throughout this paper as synonyms for the plate-bending and the plane-elasticity problem, respectively.

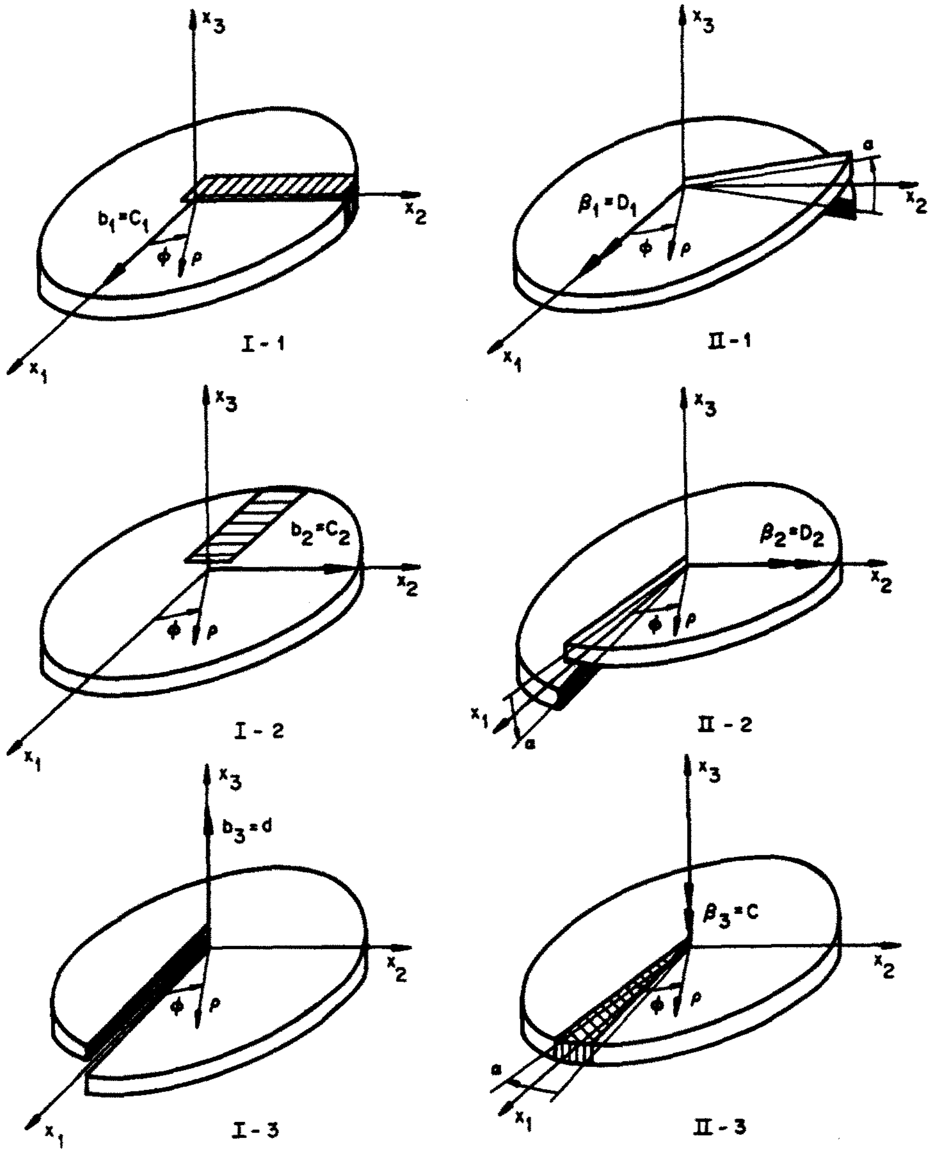


Fig. 9. Volterra's dislocations (I) and disclinations (II) as the basic geometrical singularities for the slice problem and the plate problem, respectively [cp. Fig. 3(b)].

metrical singularities for the slice problem, whereas the screw dislocation with Burgers vector b_3 and the twist disclination with Frank vector β_1, β_2 are the basic geometrical singularities for the plate problem [for comparison: the basic *statical* singularity for the slice problem is the single force $R_i = (R_1, R_2)$ acting in the middle plane x_1x_2 (cp. Fig. 8), and for the plate problem the single force F acting perpendicularly to the middle plane].

In the following we will, for a better distinction, denote the plate deflection u_3 by the letter w and reserve the letter $u_i = (u_1, u_2)$ for the displacement vector in the slice problem:

$$\text{slice problem:} \quad \text{displacement } u_i \quad (i = 1, 2), \tag{3.6-1}$$

$$\text{plate problem:} \quad \text{deflection } u_3 = w. \tag{3.6-2}$$

Furthermore, it is convenient for the following calculations to introduce new names also for the vectors describing the considered dislocations and disclinations, respec-

Table 2. New names for the Burgers vector b_i, the Frank vector β_i, and for the wedge and twist disclination dipoles β_{ij}

slice problem		plate problem	
wedge disclination	$\beta_3 = C$	twist disclination	$\begin{cases}\beta_1 = D_1\\ \beta_2 = D_2\end{cases}$
edge dislocation	$\begin{cases}b_1 = C_1\\ b_2 = C_2\end{cases}$	screw dislocation	$b_3 = d$
wedge discl. dipole	$\begin{cases}\beta_{13} = -C_2\\ \beta_{23} = +C_1\end{cases}$	twist discl. dipole	$\begin{cases}\beta_{12} = d_{12}\\ \beta_{21} = d_{21}\end{cases}$

tively, and for the tensors describing wedge and twist disclination dipoles; they are listed in Table 2.

3.2. *The displacement caused by a basic geometrical singularity*

The fundamental solutions for the infinite isotropic slice and plate subjected to a dislocation or a disclination, respectively, have already been discussed in detail by Volterra himself[6] (and [10, 18, 23]). In our context we are interested [cp. eqn (3.6) and Table 2] in the following cases:

Case 1—*slice problem:* the displacement vector $u_i = (u_1, u_2)$ caused by the wedge disclination C and the edge dislocation $C_j = (C_1, C_2)$.

Case 2—*plate problem:* the plate deflection w caused by the twist disclination $D_j = (D_1, D_2)$ and the screw dislocation d. Again we write the corresponding influence functions in the practical form suggested by Heise[24]: e.g. for the slice problem the influence function $(uC)_{i.j}$ depicts the influence of the singularity C_j on the displacement u_i

$$u_i = (uC)_{i.j} \cdot C_j \qquad (i, j = 1, 2). \tag{3.7a}$$

Because u_i and C_j are vectors, $(uC)_{i.j}$ is obviously a tensor of second rank. It depends both on the coordinates $\bar{x}_t = (\bar{x}_1, \bar{x}_2)$ of the source point, i.e. the point at which the

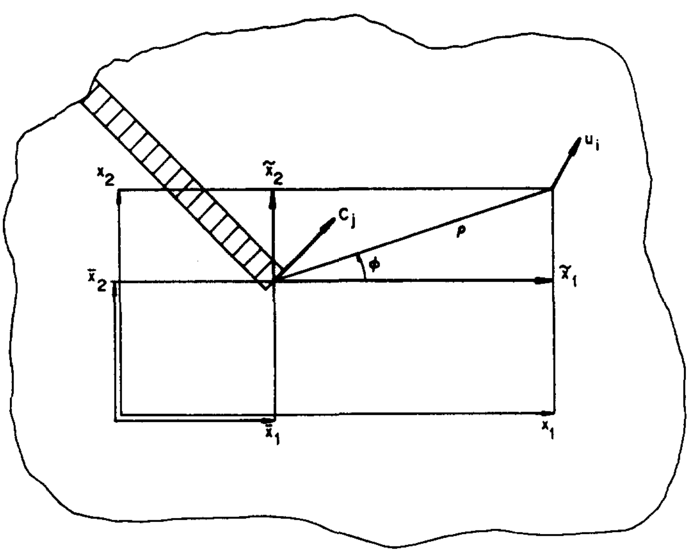


Fig. 10. Displacement u_i in the field point x_t caused by an edge dislocation C_j in the source point $\bar{x}_t$.

Table 3. The four basic influence functions for the slice and plate, respectively, subjected to a basic geometrical singularity ($i, j = 1, 2$)

slice problem	plate problem
wedge disclination C: $u_i = (uC)_{i.} \cdot C$	twist disclination D_j: $w = (wD)_{.j} \cdot D_j$
edge dislocation C_j: $u_i = (uC)_{i.j} \cdot C_j$	screw dislocation d: $w = (wd)_{.} \cdot d$

singularity (here C_j) is acting, and on the coordinates $x_t = (x_1, x_2)$ of the field point, i.e. the point at which the corresponding state variable (here u_i) is sought; thus eqn (3.7a) is written more precisely in the form

$$u_i(x_t) = (uC)_{i.j}(x_t, \bar{x}_t) \cdot C_j(\bar{x}_t) \qquad (i, j, t = 1, 2). \tag{3.7b}$$

In Fig. 9 we had chosen the source point coordinates $\bar{x}_t = 0$, i.e. the singularity was applied at the origin of the x_1x_2-system: in the following we choose $\bar{x}_t \neq 0$ (see Fig. 10), and denote the difference of the radius vectors of the field point and the source point by $\tilde{x}_t$, and the distance between these two points by ρ

$$\tilde{x}_t = x_t - \bar{x}_t, \qquad \rho^2 = \tilde{x}_t\tilde{x}_t \qquad (t = 1, 2). \tag{3.8}$$

With the six basic geometrical singularities presented, respectively, for the slice problem and the plate problem, we have to consider the following four basic influence functions as presented in Table 3. In considering these influence functions, we confine ourselves to isotropic material, with $m = 1/\nu$ being the reciprocal value of Poisson's ratio ν. In the following, all indices, if not defined differently, will take the values 1 and 2.

3.2a *Slice problem: the displacement due to a wedge disclination and to an edge dislocation.* The displacement vector $u_i(x_t)$ for the infinite isotropic slice subjected to a wedge disclination $C(\bar{x}_t)$ is given by the influence function (cp. Heise[24], in whose notation ϑ equals $-C = -\beta_3$)

$$(uC)_{i.} = \frac{1}{4\pi m}\{(m - 1)\tilde{x}_i \ln \rho - 2me_{ir}\tilde{x}_r\phi\} \tag{3.9a}$$

with $m = 1/\nu$ and the permutation tensor e_{ir} ($e_{12} = -e_{21} = 1$, $e_{11} = e_{22} = 0$); written separately for the components u_1 and u_2 eqn (3.9a) yields

$$\begin{aligned} u_1 &= \frac{C}{4\pi m}\{(m - 1)\tilde{x}_1 \ln \rho - 2m\tilde{x}_2\phi\}, \\ u_2 &= \frac{C}{4\pi m}\{(m - 1)\tilde{x}_2 \ln \rho + 2m\tilde{x}_1\phi\}. \end{aligned} \tag{3.9b}$$

The corresponding discontinuity of the displacement, jumping for Δu_i with every Burgers circuit S, is according to eqn (2.5a)

$$(\Delta uC)_{i.} = \oint_{(S)} (\partial(uC)_{i.}/\partial s)\, ds \tag{3.10}$$

$$= (uC)_{i.}\,(\phi = 2\pi) - (uC)_{i.}\,(\phi = 0), \tag{3.11}$$

i.e.

$$(\Delta uC)_{i.} = -e_{ir}\bar{x}_r \tag{3.12a}$$

or written in the form of eqn (3.9b)

$$\Delta u_1 = -C\cdot\bar{x}_2, \qquad \Delta u_2 = +C\cdot\bar{x}_1. \tag{3.12b}$$

This result is in accordance with eqn (3.5-1) (with $b_1 = b_2 = 0$ and $\beta_3 = C$) and can easily be verified from Fig. 9 (II-3) for the special case $\bar{x}_i = x_i$.

The second basic influence function for the slice problem (see Table 3) describes the influence of an edge dislocation $C_j(\bar{x}_t)$ on the displacement vector $u_i(x_t)$:

$$(uC)_{i.j} = \frac{1}{4\pi m}\{(m - 1)e_{ij}\ln\rho + 2m\delta_{ij}\phi + (m + 1)e_{jr}\bar{x}_i\bar{x}_r/\rho^2\} \tag{3.13a}$$

with the identity tensor δ_{ij} ($\delta_{11} = \delta_{22} = 1$, $\delta_{12} = \delta_{21} = 0$), or, again written separately for the components u_1 and u_2,

$$\begin{aligned} u_1 &= \frac{C_1}{4\pi m}\{2m\phi + (m + 1)\bar{x}_1\bar{x}_2/\rho^2\} + \frac{C_2}{4\pi m}\{(m - 1)\ln\rho - (m + 1)\bar{x}_1^2/\rho^2\}, \\ u_2 &= \frac{C_1}{4\pi m}\{-(m - 1)\ln\rho + (m + 1)\bar{x}_2^2/\rho^2\} + \frac{C_2}{4\pi m}\{2m\phi - (m + 1)\bar{x}_1\bar{x}_2/\rho^2\}. \end{aligned} \tag{3.13b}$$

Analogously to eqn (3.11), the corresponding discontinuity Δu_i of the displacement is found as

$$(\Delta uC)_{i.j} = (uC)_{i.j}(\phi = 2\pi) - (uC)_{i.j}(\phi = 0), \tag{3.14}$$

i.e.

$$(\Delta uC)_{i.j} = \delta_{ij}, \tag{3.15a}$$

or

$$\Delta u_1 = +C_1, \qquad \Delta u_2 = +C_2, \tag{3.15b}$$

in accordance with eqn (3.5-1) (with $b_1 = C_1$, $b_2 = C_2$ and $\beta_3 = C = 0$) and Fig. 9 [(I-1), (I-2)].

Now, as was discussed in Chapter 2.4 [cp. eqn (2.7) and Table 1], any edge dislocation C_j can be interpreted as a dipole of a wedge disclination C, the distance vector $\delta\bar{x}_i$ of which has been rotated through $-90°$ around the $+x_3$-axis. Indeed, if we differentiate the influence function $(uC)_{i.}(x_t, \bar{x}_t)$ from eqn (3.9a) with respect to the source point coordinates $\bar{x}_t$ [cp. eqns (3.7b) and (3.8)], thus getting the influence function $(uC)_{i.r}(x_t, \bar{x}_t) = \bar{\partial}_r(uC)_{i.}(x_t, \bar{x}_t)$ for the displacement $u_i(x_t)$ due to a wedge disclination *dipole* $C_r(\bar{x}_t)$, and multiply the result with the permutation tensor e_{jr}, thus rotating the dipole vector C_r through $-90°$ in the x_1x_2-plane, we obtain the equality

$$(uC)_{i.j}(x_t, \bar{x}_t) = e_{jr}\bar{\partial}_r(uC)_{i.}(x_t, \bar{x}_t). \tag{3.16}$$

The displacement $u_i(x_t)$ due to geometrical dipole singularities of higher order, e.g. an edge dislocation dipole $c_{jk}(\bar{x}_t)$, is calculated analogously, i.e. by differentiation with respect to the source point coordinates $\bar{x}_t$, e.g.

$$(uc)_{i.jk}(x_t, \bar{x}_t) = \bar{\partial}_j(uC)_{i.k}(x_t, \bar{x}_t). \tag{3.17}$$

Taking into account eqns (3.16) and (3.17), the wedge disclination C can be considered as the *ancestral* geometrical singularity for the slice problem. With $(uC)_{i.}$ once known, all other influence functions for any geometrical singularity (of Volterra type†) can be derived: by differentiation with respect to the *source* point coordinates $\bar{x}_t$, we obtain $u_i(x_t)$ due to the edge dislocation $C_j(\bar{x}_t)$, the edge dislocation dipole $c_{jk}(\bar{x}_t)$, the edge dislocation quadrupole $c_{jkl}(\bar{x}_t)$ and so forth; by differentiation of $(uC)_{i.}(x_t, \bar{x}_t)$ with respect to the *field* point coordinates x_t, we obtain the distorsion tensor $U_{hi}(x_t) = \partial_h u_i(x_t)$ due to the wedge disclination $C(\bar{x}_t)$

$$(UC)_{hi.}(x_t, \bar{x}_t) = \partial_h(uC)_{i.}(x_t, \bar{x}_t), \tag{3.18}$$

and thus the corresponding strain tensor

$$E_{hi} = \tfrac{1}{2}(U_{hi} + U_{ih}) \tag{3.19}$$

and finally, by applying Hooke's generalized law, the h-fold stress tensor

$$\Pi_{fg} = C_{fghi}E_{hi}\cdot h \tag{3.20}$$

(thickness h of the slice).

3.2b *Plate problem: the deflection due to a twist disclination and to a screw dislocation.* The plate deflection $w(x_t)$ for the infinite isotropic plate subjected to a twist disclination $D_j(\bar{x}_t)$ is given by the influence function[27]

$$(wD)_{,j} = \frac{1}{4\pi m}\{-(m + 1)\bar{x}_j \ln \rho + 2me_{jr}\bar{x}_r\phi\} \tag{3.21a}$$

or

$$w = \frac{D_1}{4\pi m}\{-(m + 1)\bar{x}_1 \ln \rho + 2m\bar{x}_2\phi\} + \frac{D_2}{4\pi m}\{-(m + 1)\bar{x}_2 \ln \rho - 2m\bar{x}_1\phi\}. \tag{3.21b}$$

The corresponding discontinuity Δw is [cp. the analogous eqns (3.9)–(3.12) for the slice problem]

$$(\Delta wD)_{,j} = (wD)_{,j}\,(\phi = 2\pi) - (wD)_{,j}\,(\phi = 0), \tag{3.22}$$

i.e.

$$(\Delta wD)_{,j} = e_{jr}\bar{x}_r \tag{3.23a}$$

or

$$\Delta w = +D_1\cdot\bar{x}_2 - D_2\cdot\bar{x}_1; \tag{3.23b}$$

this is in accordance with eqn (3.5-2) (with $b_3 = 0$ and $\beta_1 = D_1$, $\beta_2 = D_2$) and easily to be seen from Fig. 9 [(II-1), (II-2)].

† A geometrical singularity of non-Volterra type for the slice problem is, e.g. Rieder's singularity (see [24, 26]).

By differentiating $(wD)_{.j}(x_t, \bar{x}_t)$ with respect to the source point coordinates $\bar{x}_t$

$$(wd)_{jk}(x_t, \bar{x}_t) = \bar{\partial}_j(wD)_{.k}(x_t, \bar{x}_t) \quad (3.24)$$

eqn (3.21) yields the plate deflection $w(x_t)$ due to a twist disclination dipole $d_{jk}(\bar{x}_t)$

$$w = \frac{d_{jk}}{4\pi m}\{+(m+1)\,\delta_{jk}\ln\rho + 2me_{jk}\phi - (m-1)\bar{x}_j\bar{x}_k/\rho^2\}. \quad (3.25)$$

If we now set, following the considerations of Chapter 2.4 [see eqn (2.8), Tables 1 and 2],

$$d_{jk} = e_{jk}d/2 \quad (3.26)$$

in eqn (3.25), we arrive at the influence function (wd) describing the plate deflection $w(x_t)$ due to a screw dislocation $d(\bar{x}_t)$ with the Burgers vector $d = b_3$ pointing into the positive x_3-direction; we simply obtain

$$(wd) = d\cdot\frac{\phi}{2\pi} \quad (3.27)$$

with the discontinuity

$$(\Delta wd) = (wd)\,(\phi = 2\pi) - (wd)\,(\phi = 0), \quad (3.28)$$

i.e.

$$\Delta w = d \quad (3.29)$$

[cp. eqn (3.5-2) and see Fig. 9 (I-3)].

Of course the remarks at the end of the last chapter also hold true for the plate problem, and here the twist disclination D_j can be considered as the *ancestral* geometrical singularity.

3.3. *Slab analogy: the ancestral geometrical and statical singularities for the slice problem and the plate problem*

The analogy between Airy's stress function χ, used in the solution of slice problems, and the plate deflection w was already recognized by Michell[1]. Further investigations (see, e.g. [28]) showed that a one-to-one analogy, known as the slab analogy, can be established between all quantities describing the slice and the plate problem respectively, i.e. state variables, singularities and influence functions. In our context we will only give a short comparison between the presented basic geometrical singularities for slice and plate, respectively, and the corresponding basic statical singularities:

Case 1—slice problem: The basic geometrical state variable is the displacement vector u_i, and, as was shown in 3.2a, the ancestral geometrical singularity is the wedge disclination C; with the corresponding influence function connecting these two geometrical quantities $(uC)_{i.}$ [see eqn (3.9)] once known, the influence functions for the edge dislocation C_j, its dipole c_{jk}, etc., can easily be derived. The basic statical state variable is Airy's stress function χ, and the ancestral statical singularity is the single force R_j; the influence function $(\chi R)_{.j}$ plays the same role for the statical quantities as does $(uC)_{i.}$ for the geometrical quantities.

Case 2—plate problem: The basic geometrical state variable is the plate deflection w, and the ancestral geometrical singularity is the twist disclination D_j (see 3.2b); thus $(wD)_{.j}$ is the basic influence function for the geometrical quantities. The basic statical state variable is Schaefer's stress function vector γ_i (from which the moments are

Table 4. The analogy between the ancestral singularities of slice and plate

slice problem	plate problem
wedge disclination: +C	+F : single force
single force : $+R_j$	$-D_j$: twist disclination

derived by differentiation[28]), and the ancestral statical singularity is the single force F perpendicular to the middle plane (defined as positive if it has the same direction as $+x_3$ and $+w$, respectively); thus $(\gamma F)_{i.}$ is the basic influence function for the statical quantities and corresponds to $(wD)_{.j}$.

Now the slab analogy shows that geometrical (statical) quantities of the slice problem are analogous to statical (geometrical) quantities of the plate problem, and vice versa; in particular the ancestral singularities for slice and plate correspond to each other, as shown in Table 4. The change of the sign in the second line of Table 4 is of course due to the convention for D_j used in this paper (see the definition of the Frank vector in Chapter 2.1). Furthermore, between the aforementioned four basic influence functions, i.e. $(uC)_{i.}$ and $(\chi R)_{.j}$ for the slice problem and $(wD)_{.j}$ and $(\gamma F)_{i.}$ for the plate problem, the following two relationships hold:

$$+(uC)_{i.}(x_t, \bar{x}_t) \triangleq +(\gamma F)_{i.}(x_t, \bar{x}_t), \tag{3.30}$$

$$+(wD)_{.j}(x_t, \bar{x}_t) \triangleq -(\chi R)_{.j}(x_t, \bar{x}_t), \tag{3.31}$$

where the $\triangleq$ symbol means tht the influence functions become identical after simply substituting, in the isotropic case, $+m = 1/\nu$ by $(-m)$.

3.4. *Extension for anisotropic material*

For an uniformly anisotropic-elastic material all considerations, especially those of the last Chapter 3.3, still hold true; only the influence functions themselves now have a more complicated structure (see [29–33]). E.g. the deflection w due to a twist disclination D_j in an infinite plate of anisotropic material [cp. eqn (3.21) for the isotropic case] is given by the influence function

$$(wD)_{.j} = \frac{1}{2\pi} \sum_{\lambda=1}^{2} \{e_{it}\bar{x}_t(V_{ij\lambda} \ln R_\lambda + W_{ij\lambda} \arctan P_\lambda)\} \tag{3.32}$$

with the geometrical quantities

$$\begin{aligned} R_\lambda^2 &= (\bar{x}_1 + A_\lambda \bar{x}_2)^2 + (B_\lambda \bar{x}_2)^2, \\ P_\lambda &= (B_\lambda \bar{x}_2)/(\bar{x}_1 + A_\lambda \bar{x}_2), \end{aligned} \tag{3.33}$$

wherein A_λ, B_λ and the second-order tensors $V_{ij\lambda}, W_{ij\lambda}$ are constants describing the anisotropic character of the material; when approaching the isotropic case with $A_\lambda \to 0$ and $B_\lambda \to 1$, we obtain $R_\lambda \to \rho$ and $\arctan P_\lambda \to \phi$, and eqn (3.32) changes into eqn (3.21).

As in the case of isotropic material [cp. eqn (3.31)], the influence function $(wD)_{.j}$ is analogous to $(\chi R)_{.j}$, i.e. when we substitute A_λ, B_λ and $V_{ij\lambda}$, $W_{ij\lambda}$ in eqns. (3.32) and (3.33) by the corresponding constants α_λ, β_λ and $v_{ij\lambda}$, $w_{ij\lambda}$, we directly obtain $(\chi R)_{.j}$ for the infinite slice of the same anisotropic material.

4. CONCLUSIONS

Twist and wedge disclinations together with edge and screw dislocations have to be considered as the *basic geometrical singularities* in plane-elasticity and plate-bending problems. From the point of view of potential theory, the *ancestral geometrical singularity* for the slice problem is the wedge disclination, and for the plate problem the twist disclination; with the influence functions for these two types of Volterra's "distorsioni" once known, the corresponding influence functions for all geometrical singularities of higher order are obtained by differentiation with respect to the source point coordinates. In the same sense the *ancestral statical singularity* for the slice and plate problem, respectively, is the single force.

A systematic classification of the geometrical and statical singularities and state variables and of the related influence functions can be very helpful for a better understanding of plane-elasticity and plate-bending problems and their solution.

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